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EXAMPLE OF SOLVING TRANSONIC EQUATIONS FOR A SHOCK-FREE FLOW PAST A SYMMETRIC PROFILE*

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A parametric method /1/ of solving the transonic Kármán– Fal'kovich equations is developed. The nozzle solution is generalized to the case of the flows not symmetric about the longitudinal axis of the nozzle. A procedure of passing from this solution to the case of a flow past a profile discussed in /2/ is then shown. This in fact means that the real and imaginary part of the complex function describing this flow have been obtained. The resulting solution depends on three constants determining the dimensions of the profile (length of chord and the maximum thickness) and also the flow rate at infinity. Numerical analysis is used to obtain the condition for the flow to be shock-free, and a continuous velocity field is constructed under the conditions close to the limiting state. Setting up a flow chart for the cases when the condition of no shock is violated shows that a three-sheeted fold appears, the top of which lies within the supersonic region. This confirms the conclusion made in /3–5/ that in a typical case of a flow past a profile, the shock wave originates not at the sonic stream line, but within the zone. The example constructed can be used as the basis for the theory of flow past a profile of a sufficiently general form, of a gas stream subsonic at infinity. Below the transonic Tricomi model is used to show the corresponding generalization.

1. Let us consider an approximate system of transonic equations

$$uu_x = v_y, \quad u_y = v_x \tag{1.1}$$

where u and v are reduced dimensionless velocities of perturbation of a homogeneous sonic flow and x, y are the Cartesian coordinates. We shall show one exact solution of the system (1.1)

$$\begin{gathered} u = -4/(3C_1^2p^2) + C_1^2t^2/4 + C_1\lambda p/2 \\ v = -C_1^3t^3/12 - 4t/(3C_1p^2) - C_1^2\lambda pt/4, \quad x = -4/(3C_1^3p^2) - C_1t^2/4 - \lambda p, \quad y = t \end{gathered} \tag{1.2}$$

Here p and t are parameters, and C_1, λ are arbitrary constants. The solution describes the motion in plane Laval nozzles with local supersonic zones at the walls, obtained in /1/. Let us write (1.2) in the symbolic form

$$u = u(p, t; C_1, \lambda), \ v = v(p, t; C_1, \lambda), \ x = x(p, t; C_1, \lambda), \ y = y(p, t; C_1, \lambda) \tag{1.3}$$

Using the first two equations of (1.3), we write p and t in terms of u, v, C_1, λ to obtain

$$p = P(u, v, C_1, \lambda), \ t = T(u, v; C_1, \lambda)$$

and this yields

$$\begin{gathered} x = x(P(u, v; C_1, \lambda), T(u, v; C_1, \lambda), C_1, \lambda) = X(u, v; C_1, \lambda) \\ y = y(P(u, v; C_1, \lambda), T(u, v; C_1, \lambda), C_1, \lambda) = Y(u, v; C_1, \lambda) \end{gathered} \tag{1.4}$$

The functions X and Y satisfy the linear system

$$uY_v = X_u, \ Y_u = X_v \tag{1.5}$$

which is equivalent to (1.1). Obviously, differentiating and integrating X and Y with respect to C_1, λ and v leads to new solutions of (1.5).

Let us introduce the following generalized differentiation operator

$$\frac{\partial^{m, n, k}}{\partial C_1^m \partial\lambda^n \partial v^k} \tag{1.6}$$

defined for the integral values of m, n and k (the positive values denote differentiation, and the negative values the integration). Applying the operator (1.6) written in parametric form to (1.2), yields more new solutions.

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The solution (1.4) equivalent to (1.2) has a singularity at the point $u = u_0 = -(^3/_2\lambda)^{2/3}$, $v = 0$. The singularity is a branch point of the order $^1/_2$ for the functions X and Y /2/. We note two properties of the operator (1.6): a) differentiation with respect to v reduces the order of the singularity at the point $u = u_0, v = 0$ by one, and the symmetry is upset; the solution in which $v = 0$ when $y = 0$ transforms into a new solution where $u = 0$ when $y = 0$, b) differentiation with respect to λ also reduces the order of the singularity by one, but the symmetry property is preserved.

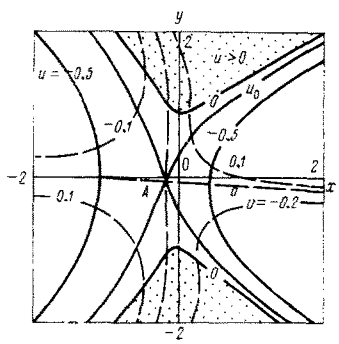

Fig.1

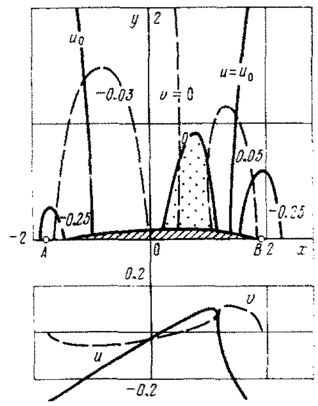

Fig.2

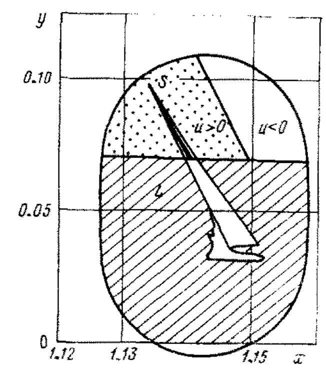

Fig.3

2. Application of the operator $\partial^{0,1,0}/\partial C_1{}^0 \partial\lambda \partial v^0$ to x and y from (1.2) yields the expression for x_1 and y_1 which, together with u and v from (1.2), determine a symmetric solution with the singularity index of $-^1/_2$. If we now integrate x_1 and y_1 with respect to v, we obtain (brackets denote the vectors)

$$(x_c, y_c) = \frac{\partial^{0,1,-1}(X, Y)}{\partial C_1^0 \partial\lambda^1 \partial v^{-1}} \tag{2.1}$$

with the same singularity as in (1.4).

A different parametrization of the nozzle solution (1.2) was used in /5/ where

$$p = ms, \quad \lambda = -C_2/m, \quad m = -i\left(\frac{2}{C_1}\right)^{3/2} 3^{-1/2} \tag{2.2}$$

and the properties of a flow generated by the solution with

$$(x_2, y_2) = \frac{\partial^{0,0,2}(X, Y)}{\partial C_1^0 \partial C_2^2 \partial v^2}$$

were studied. By virtue of the linearity of the hodograph equations (1.5) we can write, from (1.2) and (2.1), the combination (k is an arbitrary constant)

$$(x_T, y_T) = (x, y) + (x_c, y_c)\, k$$

which assumes the following form in the variables (2.2):

$$u = C_1(1 - C_2 s^3)/(2s^2) + C_1{}^2 t^2/4 \quad v = C_1{}^2 t(2 + C_2 s^3)/(4s^2) - C_1{}^3 t^2/12$$
$$x_T = (1 + 2C_2 s^3)/(2s^2) - C_1 t^2/4 + kC_1 t/(2s) \quad y_T = t + k/s \tag{2.3}$$

The solution constructed describes a class of flows in Laval nozzles nonsymmetric about the longitudinal axis of flow. Fig.1 depicts the lines $u = \text{const}$ (solid) and $v = \text{const}$ (dashed) in the $x\,y$-plane, for $k = -0.1$, $C_1 = -2^{-7}3^{-3}5^5$, $C_2 = -0.09496$. The isolines $v = 0$, $u = u_0$ intersect at the point A. The value u_0 represents the characteristic subsonic velocity of this nozzle. The actual representation for

$$(x_1, y_1) = \frac{\partial^{0,1,0}(X, Y)}{\partial C_1^0 \partial\lambda\, \partial v^0}$$

(with the accuracy of up to the multiplication factor) is obtained in the form

$$x_1 = 2s(2 + C_2 s^3 - C_1 t^2 s^2)/(C_1 K)$$
$$y_1 = -4ts^3/(C_1 K), \quad K = (2 + C_2 s^3)^2 - 8C_1 t^2 s^2 \tag{2.4}$$

where u and v are given by (2.3).

The singularity of this solution in the parameter plane is $t = 0$, $C_2s^3 = -2$, which corresponds to the infinity of the plane x_1y_1 with the same characteristic velocity u_0. The sonic line emerges from the coordinate origin ($s \to \infty$), forms a local supersonic zone and returns to the coordinate origin ($s \to 0$, $t \to \infty$). The condition $v = 0$ holds everywhere along the x-axis except at the coordinate origin where another singularity is situated. A three-sheet fold similar to that discussed in /5/ is present in the region of supersonic velocities.

To remove the ambiguous character of the velocity field in the xy-plane, we combine x_1, y_1 and the nozzle solution (1.2)

$$(x, y) = (x_1, y_1) + (x, y)D \tag{2.5}$$

The flow described by (2.5) together with u and v from (1.2) is subsonic when $D < -(4C_2C_1)^{-1}$. The relation $v = 0$ holds everywhere along the x-axis except on the segment AB shown in Fig.2. The part of the x-axis contained between A and B is obtained from the condition $y = 0$, $v \neq 0$ and

$$t^2 = [(2 + C_2s^3)^2 = 4s^3/(DC_1)]/(6C_1s^2) \tag{2.6}$$

When $t \to 0$, (2.6) yields the coordinates s_1 and s_2 of the points corresponding to A and B. A part of the first quadrant of the st parameter plane bounded by the segment of the s-axis contained between s_1 and s_2 and by the curve (2.6), corresponds to the flow outside the cut AB, and $t = 0$, $C_2s^3 = -2$. again corresponds to the infinity on the xy-plane where $v = 0$ and $u = u_0$.

When the values of C_1 and C_2 are as before and $-0{,}81 < D \leqslant 0$, the structure of the local supersonic zone is identical to that occurring at $D = 0$. when $-1.95 < D \leqslant -0.81$, one of the branches of the limiting line vanishes and the other, with the cusp within the zone, approaches the sonic line without however touching it. Finally, when $D = -1.95$, no limiting lines appear within the zone. Fig.2 depicts the velocity field at $D = -1.95$ and the stream line obtained by integrating the equation $dy/dx = v$ from A to B and regarded as the generatrix of the profile. The velocity distribution along the cut AB is also shown.

Study of the flows in the cases when the condition of no shock is violated may yield some information about the mechanism of shock formation. Fig.3 depicts a part of the flow for $D = -1.88$, with C_1 and C_2 remaining the same. The characteristic curves of one family are reflected from the sonic line and, beginning from the point S, intersect with themselves thus forming a fold. If the dimensions of such fold are small, then we can construct separately the equipotential stream lines of the first and third sheet and the line along which the shock polar is formed, and show that on approaching the point S these lines coincide in the limit with each other and remain sufficiently close at some distance from S. Assuming a certain admissible error, we can regard such a split shock as a model of a weak shock wave shown in Fig.3 as the line SL of varying thickness.

3. In order to construct a solution to the problem of a flow past a profile of sufficiently general shape, we can use the following set of partial solutions singular at the point $v = 0$, $u = u_0$ and obtained from (1.2):

$$(x_{-n}, y_{-n}) = \frac{\partial^{0,-n,0}(X, Y)}{\partial C_1^0 \partial\lambda^{-n} \partial v^0} \quad (n = 1, 2, \ldots) \tag{3.1}$$

A single integration ($n = 1$) yields

$$x_{-1} = C_1^3t^2(4 - C_2s^3)/(16s^3) + C_1^2(10C_2^2s^6 - 5C_2s^3 + 4)/(40s^5)$$
$$y_{-1} = (2 + C_2s^3)C_1^2t/(4s^3)$$

and repeated integration ($n = 2$) yields the expressions

$$x_{-2} = (C_1^4t^2s^3(-27C_1^3t^6s^6/8 + 9C_1^3t^4s^4(1 + 3C_2s^3) + 5C_1t^2s^2(3 - 6C_2s^3 - 15C_2^2s^6) + 20(5 + 21C_2s^3 - 3C_2^2s^6 + 4C_2^3s^9)) + C_1^3(10 + 152C_2s^3 - 300C_2^2s^6 + 320C_2^3s^9 - 20C_2^4s^{12}))/(1280s^8)$$
$$y_{-2} = C_1^3t(27C_1^3t^6s^6/7 - 9C_1^2t^4s^4(2 + 3C_2s^3) + 60C_1t^2s^2(C_2^2s^6 + C_2s^3 + 1) - 20(C_2s^3 + 2)(2C_2^2s^6 - 10C_2s^3 - 1))/(640s^6)$$

The solutions obtained can be combined with the regular solutions such as e.g. Chaplygin solutions of the Tricomi equation. Another possible method of increasing the generality of the new solutions consists of passing in (1.2) from the constants C_1, λ to $C_1(C_1^*, \lambda^*), \lambda(C_1^*, \lambda^*)$ and differentiating with respect to C_1^*, λ^*.

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